Δείξτε ότι $\forall \alpha, \beta \in \Re,\ n \in /N$

$$(\alpha+\beta)^n = \alpha^n + \binom{n}{1}\alpha^{n-1}\beta + \binom{n}{2}\alpha^{n-2}\beta^2 + \dots + \binom{n}{n-1}\alpha\beta^{n-1} + \beta^n$$

ή ισοδύναμα (με $\alpha^0=\beta^0=1$)

$$(\alpha+\beta)^n = \sum_{\kappa=0}^{n}\binom{n}{\kappa}\alpha^{n-\kappa}\beta^{\kappa}$$

Απόδειξη

Η απόδειξη θα γίνει με επαγωγή.

Για $n=1$ ισχύει $(\alpha+\beta)^1 = \alpha+\beta$

Έστω ότι για $n=\nu$ ισχύει η σχέση

$$(\alpha+\beta)^\nu = \sum_{\kappa=0}^{\nu}\binom{\nu}{\kappa}\alpha^{\nu-\kappa}\beta^{\kappa}$$

Θα δείξω ότι ισχύει η ίδια σχέση και για $n=\nu+1$.

Δηλαδή $(\alpha+\beta)^{\nu+1} = \sum_{\kappa=0}^{\nu+1}\binom{\nu+1}{\kappa}\alpha^{\nu+1-\kappa}\beta^{\kappa}$

Πράγματι $(\alpha+\beta)^{\nu+1} = (\alpha+\beta)^{\nu}\cdot(\alpha+\beta)$

$$= \sum_{\kappa=0}^{\nu}\binom{\nu}{\kappa}\alpha^{\nu-\kappa}\beta^{\kappa}\cdot(\alpha+\beta)$$

$$= \alpha\cdot\sum_{\kappa=0}^{\nu}\binom{\nu}{\kappa}\alpha^{\nu-\kappa}\beta^{\kappa} + \beta\cdot\sum_{\kappa=0}^{\nu}\binom{\nu}{\kappa}\alpha^{\nu-\kappa}\beta^{\kappa}$$

$$= \sum_{\kappa=0}^{\nu}\binom{\nu}{\kappa}\alpha^{\nu+1-\kappa}\cdot\beta^{\kappa} + \sum_{\kappa=0}^{\nu}\binom{\nu}{\kappa}\alpha^{\nu-\kappa}\cdot\beta^{\kappa+1}$$

$$= \sum_{\kappa=0}^{\nu}\binom{\nu}{\kappa}\alpha^{\nu-(\kappa-1)}\beta^{\kappa} + \sum_{\kappa=0}^{\nu}\binom{\nu}{\kappa}\alpha^{\nu-\kappa}\beta^{\kappa+1}$$

↓ θέτω $\lambda=\kappa+1$

$$=\sum_{\kappa=0}^{\nu}\binom{\nu}{\kappa}\alpha^{\nu-(\kappa-1)}\beta^{\kappa}+\sum_{\lambda=1}^{\nu+1}\binom{\nu}{\lambda-1}\alpha^{\nu-(\lambda-1)}\beta^{\lambda}$$

$\downarrow$ θέτω $\kappa=\lambda$

$$=\sum_{\lambda=0}^{\nu}\binom{\nu}{\lambda}\alpha^{\nu-(\lambda-1)}\beta^{\lambda}+\sum_{\lambda=1}^{\nu+1}\binom{\nu}{\lambda-1}\alpha^{\nu-(\lambda-1)}\beta^{\lambda}$$

$$=\alpha^{\nu+1}+\sum_{\lambda=1}^{\nu}\binom{\nu}{\lambda}\alpha^{\nu-(\lambda-1)}\beta^{\lambda}+\sum_{\lambda=1}^{\nu}\binom{\nu}{\lambda-1}\alpha^{\nu-(\lambda-1)}\beta^{\lambda}+\beta^{\nu+1}$$

$$=\alpha^{\nu+1}+\sum_{\lambda=1}^{\nu}\left[\binom{\nu}{\lambda}\alpha^{\nu-(\lambda-1)}\beta^{\lambda}+\binom{\nu}{\lambda-1}\alpha^{\nu-(\lambda-1)}\beta^{\lambda}\right]+\beta^{\nu+1}$$

$$=\alpha^{\nu+1}+\sum_{\lambda=1}^{\nu}\left[\binom{\nu}{\lambda}+\binom{\nu}{\lambda-1}\right]\alpha^{\nu-(\lambda-1)}\beta^{\lambda}+\beta^{\nu+1}$$

$\downarrow$ ιδιότητα Pascal

$$\binom{n}{\kappa}+\binom{n}{\kappa-1}=\binom{n+1}{\kappa}$$

$$=\alpha^{\nu+1}+\sum_{\lambda=1}^{\nu}\binom{\nu+1}{\lambda}\alpha^{\nu+1-\lambda}\beta^{\lambda}+\beta^{\nu+1}=$$

$$=\sum_{\lambda=0}^{\nu+1}\binom{\nu+1}{\lambda}\alpha^{\nu+1-\lambda}\beta^{\lambda}\ .$$

Αυτό ολοκληρώνει την απόδειξη.

ΠΑΡΑΤΗΡΗΣΗ

Η n-οστή γραμμή του τριγώνου του Pascal μας δίνει τους συντελεστές του αναπτύγματος $(\alpha+\beta)^n$.

$(\alpha+\beta)^1 =$ $1\alpha \quad + \quad 1\beta$

$(\alpha+\beta)^2 =$ $1\alpha^2 \quad 2\alpha\beta \quad 1\beta^2$

$(\alpha+\beta)^3 =$ $1\alpha^3 \quad 3\alpha^2\beta \quad 3\alpha\beta^2 \quad 1\beta^3$

$(\alpha+\beta)^4 =$ $1\alpha^4 \quad 4\alpha^3\beta \quad 6\alpha^2\beta^2 \quad 4\alpha\beta^3 \quad 1\beta^4$

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